Pergamon

Int. J. Solids Structures Vol. 33, No. 15, pp. 2213–2223, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00049–6

A MATHEMATICAL ANALYSIS OF THE ELASTO-PLASTIC ANTI-PLANE SHEAR PROBLEM OF A POWER-LAW MATERIAL AND ONE CLASS OF CLOSED-FORM SOLUTIONS

XIN-LIN GAO†
Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2G8

(*Received* 25 *July* 1994; *in revised form* 8 *January* 1995)

Abstract—A mathematical analysis of the elasto-plastic anti-plane shear problem of a power-law hardening material with infinitesimal deformations is presented in this paper. Hencky's deformation theory and von Mises' yield criterion are used in the analysis. The formulation is facilitated by using a complex variable representation and by choosing the only non-vanishing displacement component as the basic unknown. By introducing a differential transformation, the non-linear equation system describing the problem is first reduced to a solvable system of two partial differential equations. A general solution of this equivalent system is then derived using analytic function theory. Finally, one class of closed-form solutions is obtained for the telescope shear type problem of the power-law material by applying the general solution directly. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Anti-plane shear (or longitudinal shear) problems have been well studied in the context of elasticity theory because of their simplicity. Closed-form solutions of and general discussions on such problems with infinitesimal deformations are readily available [see e.g. Milne-Thomson (1962); Eshelby (1979); Atkin and Fox (1980); Horgan and Miller (1994)]. Vigorous analyses for finite anti-plane shear problems are also rich in the literature [see e.g. Adkins (1954); Knowles (1976, 1977a,b); Gurtin and Temam (1981); Abeyaratne and Horgan (1983); Jiang and Knowles (1991); Polignone and Horgan (1992); Raymond (1993)]. However, this is not the case for elasto-plastic anti-plane shear problems. Available elasto-plastic analytical solutions are rather limited and are for stress concentration problems only. For example, Koskinen (1963) and Rice (1966) provided solutions for notched/cracked prismatic bodies of elastic–perfectly plastic materials; Neuber (1961), Rice (1967) and Amazigo (1974) presented solutions for similar bodies of strain-hardening materials. All these solutions were obtained by using the hodograph transformation technique based on implicit function theory, which makes the roles of the dependent variables and independent variables interchange and thus reduces the non-linear problems to linear ones. In addition, a solution for an infinite strain-hardening medium with a circular hole was derived by Tuba (1969) using a perturbation method. It seems, to the best of the author's knowledge, that no closed-form solution has yet been reported for an elasto-plastic anti-plane shear problem other than stress concentration problems. Also, general procedures for treating this class of elasto-plastic problems in a systematic and conventional way are lacking. Therefore, further studies are still needed.

In this paper we intend to present a general mathematical analysis of the elasto-plastic anti-plane shear problem of a power-law material with infinitesimal deformations and to derive possible closed-form solutions of such problems. In Section 2 the problem is formulated by using a complex variable representation and by choosing the only non-vanishing displacement component as the basic unknown. This formulation is based on Hencky's deformation theory and von Mises' yield criterion. The non-linear equation system describing the problem is reduced to a solvable system of two partial differential equations by

†Presently at the Department of Nuclear Engineering and Engineering Sciences, University of Wisconsin—Madison, WI 53706-1687, U.S.A.

introducing a differential transformation. A general solution of this equivalent system is then obtained using analytic function theory.

In Section 3 one family of boundary-value problems of the telescopic shear type are solved by applying the general solution directly. Consequently, one class of closed-form solutions is obtained for this type of elasto-plastic anti-plane shear problem, which include the corresponding elastic solutions as special cases.

A summary of the analytical development presented in the paper is provided in Section 4. The limitations of the present analysis are also indicated.

2. FORMULATION OF THE PROBLEM

Under the assumptions of infinitesimal deformations, isotropic hardening, monotone loading without unloading, no body forces acting and material incompressibility, the basic equations, which embody Hencky's deformation theory and von Mises' yield criterion, for a power-law hardening body under anti-plane shear are as follows:

the equilibrium equation

$$\frac{\partial \tau_{x3}}{\partial x} + \frac{\partial \tau_{y3}}{\partial y} = 0, \tag{1}$$

the geometrical equations

$$\gamma_{x3} = \frac{\partial w}{\partial x}, \quad \gamma_{y3} = \frac{\partial w}{\partial y}, \tag{2}$$

and the constitutive equations

$$\gamma_{x3} = \frac{3\varepsilon_e}{\sigma_e}\tau_{x3}, \quad \gamma_{y3} = \frac{3\varepsilon_e}{\sigma_e}\tau_{y3}, \tag{3}$$

$$\sigma_e = A\varepsilon_e^n, \tag{4}$$

where A and n are material constants, with $A > 0$ and $0 < n \leqslant 1$; and σ_e and ε_e are, respectively, the effective stress and strain, with

$$\varepsilon_e = \frac{1}{\sqrt{3}}(\gamma_{x3}^2 + \gamma_{y3}^2)^{1/2}. \tag{5}$$

In the above equations, w is the only non-vanishing displacement component, $w = w(x, y)$; τ_{x3}, τ_{y3} are the non-vanishing stress components, $\tau_{x3} = \tau_{x3}(x, y)$, $\tau_{y3} = \tau_{y3}(x, y)$; and $\gamma_{x3}/2$, $\gamma_{y3}/2$ are the non-vanishing strain components, $\gamma_{x3} = \gamma_{x3}(x, y)$, $\gamma_{y3} = \gamma_{y3}(x, y)$. For simplicity, we shall call γ_{x3} and γ_{y3} strain components instead. Also, we mention in passing that the subscript 3 in the above expressions stands for the direction normal to the x–y plane.

Equations (1)–(5) in conjunction with suitable prescribed boundary conditions define the boundary-value problem for determining the stress, strain and displacement fields in the body considered. We note that no compatibility equations appear here. The reason for this is that we choose the displacement component $w(x, y)$ as the basic unknown so that the strain components in terms of w, as given in eqn (2), are themselves compatible.

It is clear that this is a plane problem with the Cartesian coordinates (x, y) as two independent variables. Hence, we can use a complex variable representation. To this end, we introduce the complex conjugate variables $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, where $\mathrm{i} = (-1)^{1/2}$ as usual. Then it follows that

$$x = \frac{1}{2}(z+\bar{z}), \quad y = \frac{1}{2i}(z-\bar{z}). \tag{6}$$

These bijective relations (identities) allow us to use $(z, \bar{z})$ as two independent variables in place of (x, y). Using eqn (6) in eqn (2) gives

$$\gamma_{x3} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}}, \quad \gamma_{y3} = i\left(\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}}\right). \tag{7}$$

Substituting eqn (7) into eqn (5) results in

$$\varepsilon_e = \frac{2}{\sqrt{3}}\left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{1/2}. \tag{8}$$

Inserting eqn (7) into eqn (3) and using eqns (4) and (8) will yield

$$\tau_{x3} = K\left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}}\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}}\right), \quad \tau_{y3} = K\left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}} i\left(\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}}\right), \tag{9}$$

where

$$K \equiv \frac{2^{n-1}}{(\sqrt{3})^{n+1}} A \tag{10}$$

is a constant.

If we were to substitute eqn (9) into eqn (1), we would obtain the final governing equation of the problem for the unknown $w(z, \bar{z}) \equiv w(x, y)$. However, since the expressions of τ_{x3} and τ_{y3} in terms of $w(z, \bar{z})$, as given in eqn (9), are already in very complicated forms, the resulting governing equation from this intended substitution will be highly non-linear such that it is impractical to solve the equation analytically. Hence, it is desirable to simplify these expressions before the required substitution. By inspection, it is found that one possible such simplification is to introduce the following differential transformation:

$$\left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}} \frac{\partial w}{\partial z} \equiv \frac{\partial \Psi}{\partial z}, \quad \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}} \frac{\partial w}{\partial \bar{z}} \equiv \frac{\partial \Psi}{\partial \bar{z}}, \tag{11}$$

where $\Psi = \Psi(z, \bar{z})$ is an analytic function yet to be determined. With this transformation, eqn (9) reduces to

$$\tau_{x3} = K\left(\frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial \bar{z}}\right), \quad \tau_{y3} = iK\left(\frac{\partial \Psi}{\partial z} - \frac{\partial \Psi}{\partial \bar{z}}\right). \tag{12}$$

Now substituting eqn (12) into eqn (1) will give

$$\frac{\partial^2 \Psi}{\partial z \, \partial \bar{z}} = 0. \tag{13}$$

This is Laplace's equation for the unknown Ψ. As a well-studied linear equation in potential theory, its solution is readily obtainable. In this way, we have overcome the difficulty of having to solve the otherwise highly non-linear governing equation just mentioned above. However, we have yet to answer the question whether eqn (11), which defines $\Psi(z, \bar{z})$

through its first derivatives that are expressed in terms of the first derivatives of $w(z, \bar{z})$, can establish a unique relation between $\Psi(z, \bar{z})$ and $w(z, \bar{z})$. In other words, we need to find the condition under which such an analytic function $\Psi(z, \bar{z})$ exists.

From analytic function theory [see e.g. Flanigan (1972)], it follows that an analytic function may be obtained from its exact differential by integration. Hence, to explore the existence of $\Psi(z, \bar{z})$ is equivalent to finding out its exact differential $\mathrm{d}\Psi$. By definition, we have

$$\mathrm{d}\Psi = \frac{\partial \Psi}{\partial z}\mathrm{d}z + \frac{\partial \Psi}{\partial \bar{z}}\mathrm{d}\bar{z}, \tag{14}$$

while from eqn (11) it follows that

$$\frac{\partial \Psi}{\partial z}\mathrm{d}z + \frac{\partial \Psi}{\partial \bar{z}}\mathrm{d}\bar{z} = M(z, \bar{z})\,\mathrm{d}z + N(z, \bar{z})\,\mathrm{d}\bar{z}, \tag{15}$$

where

$$M(z, \bar{z}) \equiv \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}} \frac{\partial w}{\partial z}, \quad N(z, \bar{z}) \equiv \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)^{\frac{n-1}{2}} \frac{\partial w}{\partial \bar{z}}. \tag{16}$$

Note that $M\,\mathrm{d}z + N\,\mathrm{d}\bar{z}$ is an exact differential if and only if [see e.g. Boresi and Chong (1987)]

$$\frac{\partial M}{\partial \bar{z}} = \frac{\partial N}{\partial z}. \tag{17}$$

Then substituting eqn (16) into eqn (17) yields

$$\frac{\partial^2 w}{\partial z^2}\left(\frac{\partial w}{\partial \bar{z}}\right)^2 - \frac{\partial^2 w}{\partial \bar{z}^2}\left(\frac{\partial w}{\partial z}\right)^2 = 0, \tag{18}$$

with $(\partial w/\partial z)(\partial w/\partial \bar{z}) \neq 0$ [i.e. $\varepsilon_e \neq 0$ from eqn (8)].

Now note that from eqn (11) we have

$$\frac{\partial w}{\partial z} = \left(\frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial \bar{z}}\right)^{\frac{1-n}{2n}} \frac{\partial \Psi}{\partial z}, \quad \frac{\partial w}{\partial \bar{z}} = \left(\frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial \bar{z}}\right)^{\frac{1-n}{2n}} \frac{\partial \Psi}{\partial \bar{z}} \tag{19}$$

for $\partial\Psi/\partial z \neq 0$ and $\partial\Psi/\partial \bar{z} \neq 0$. Then using eqn (19) in eqn (18) will give

$$\frac{\partial^2 \Psi}{\partial z^2}\left(\frac{\partial \Psi}{\partial \bar{z}}\right)^2 - \frac{\partial^2 \Psi}{\partial \bar{z}^2}\left(\frac{\partial \Psi}{\partial z}\right)^2 = 0 \tag{20}$$

for $(\partial\Psi/\partial z)(\partial\Psi/\partial \bar{z}) \neq 0$. This is the necessary and sufficient condition for the existence of $\Psi(z, \bar{z})$, which is to be determined from its first derivatives defined in eqn (11) by integration.

Thus the original equation system defined in eqns (1)–(5) has already been transformed to the following equivalent system

$$\left.\begin{aligned} &\frac{\partial^2\Psi}{\partial z\,\partial\bar{z}} = 0, \\ &\frac{\partial^2\Psi}{\partial z^2}\left(\frac{\partial\Psi}{\partial\bar{z}}\right)^2 - \frac{\partial^2\Psi}{\partial\bar{z}^2}\left(\frac{\partial\Psi}{\partial z}\right)^2 = 0, \end{aligned}\right\} \tag{21a–b}$$

subject to the condition $(\partial\Psi/\partial z)(\partial\Psi/\partial\bar{z}) \neq 0$. A general solution of this system can now be derived.

Since eqn (21a) is a Laplace equation, any harmonic function $\Psi(x,y) \equiv \Psi(z,\bar{z})$ is a solution of it. Hence, either the real or imaginary part of any analytic function can be a solution of eqn (21a). Suppose that $f(z) = p(x,y)+\mathrm{i}q(x,y)$ is an analytic function; then it follows that

$$p(x,y) \equiv \mathrm{Re}f(z) = \tfrac{1}{2}[f(z)+\overline{f(z)}], \quad q(x,y) \equiv \mathrm{Im}f(z) = \frac{1}{2\mathrm{i}}[f(z)-\overline{f(z)}], \tag{22}$$

where Re and Im denote, respectively, the real and imaginary parts, and the superposed bar denotes the conjugate of the function.

Now we choose $q(x,y)$ to be a solution of eqn (21a), i.e.

$$\Psi(z,\bar{z}) = \frac{1}{2\mathrm{i}}[f(z)-\overline{f(z)}]. \tag{23}$$

Then it follows that

$$\left.\begin{aligned} &\frac{\partial\Psi}{\partial z} = \frac{1}{2\mathrm{i}}f'(z), \quad \frac{\partial\Psi}{\partial\bar{z}} = -\frac{1}{2\mathrm{i}}\overline{f'(z)}, \\ &\frac{\partial^2\Psi}{\partial z^2} = \frac{1}{2\mathrm{i}}f''(z), \quad \frac{\partial^2\Psi}{\partial\bar{z}^2} = -\frac{1}{2\mathrm{i}}\overline{f''(z)}, \quad \frac{\partial^2\Psi}{\partial z\,\partial\bar{z}} = 0. \end{aligned}\right\} \tag{24a–e}$$

Equation (24e) implies that eqn (21a) is automatically satisfied. Substituting eqns (24a–d) into eqn (21b) will yield

$$f''(z)[\overline{f'(z)}]^2+\overline{f''(z)}[f'(z)]^2 = 0. \tag{25}$$

Clearly, $f'(z) = 0$ (for some z) is a trivial solution of this equation. Note that if an analytic function vanishes in part of a domain, then it is zero in the entire domain [see e.g. Muskhelishvili (1953)]. Then this solution gives $f'(z) \equiv 0$ for all z, and thus it follows that $f(z) \equiv$ constant in the whole domain, which is of no interest. Hence, we exclude this trivial solution by assuming $f'(z) \neq 0$. In fact, this is also required by the condition $(\partial\Psi/\partial z)(\partial\Psi/\partial\bar{z}) \neq 0$, as seen from eqns (24a,b). With $f'(z) \neq 0$, it then follows from eqn (25) that

$$\frac{f''(z)}{[f'(z)]^2} = -\frac{\overline{f''(z)}}{[\overline{f'(z)}]^2}. \tag{26}$$

Note that

$$G(z) \equiv \frac{f''(z)}{[f'(z)]^2} = R(x,y)+\mathrm{i}S(x,y); \quad R(x,y) \equiv \operatorname{Re} G(z), \quad S(x,y) \equiv \operatorname{Im} G(z) \tag{27}$$

is also analytic for any z satisfying $f'(z) \neq 0$. Then we have the Cauchy–Riemann conditions

$$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y}, \quad \frac{\partial R}{\partial y} = -\frac{\partial S}{\partial x}. \tag{28}$$

Now from eqns (27) and (26) we obtain

$$R(x,y) = 0, \tag{29a}$$

$$G(z) = \mathrm{i}S(x,y). \tag{29b}$$

Then using eqns (29a,b) in eqn (28) gives

$$S(x,y) = -C, \tag{30}$$

where C ($\neq 0$) is a real constant. Thus it follows from eqns (27), (29b) and (30) that

$$\frac{f''(z)}{[f'(z)]^2} = -\mathrm{i}C. \tag{31}$$

Integrating eqn (31) twice will give

$$f(z) = \frac{1}{\mathrm{i}C}\ln(z+D)+E, \tag{32}$$

where D and E are two complex constants. Thus combining eqns (23) and (32) yields

$$\Psi(z,\bar{z}) \equiv \Psi(x,y) = \operatorname{Im} f(z), \tag{33a}$$

$$f(z) = \frac{1}{\mathrm{i}C}\ln(z+D)+E. \tag{33b}$$

Clearly, we have from eqns (33a,b) that $(\partial\Psi/\partial z)(\partial\Psi/\partial\bar{z}) \neq 0$ and $f'(z) \neq 0$ for any finite domain, as expected. Hence, eqns (33a,b) give a general solution of the equivalent system defined in eqns (21a,b). In other words, any such analytic function $f(z)$ can solve the equivalent system and thus the original system exactly. Similarly, by choosing $p(x,y)$ to be a solution of eqn (21a) we find that

$$\Psi(z,\bar{z}) = \operatorname{Re} f(z), \quad f(z) = \frac{1}{C_1}\ln(z+D_1)+E_1 \tag{34}$$

is also a general solution of the equivalent system, where C_1 ($\neq 0$) is a real constant and D_1 and E_1 are two complex constants. It is apparent that these two solutions are of the same type. Hence, we do not call them two general solutions.

Now we are in a position to derive the expressions for the stress, strain and displacement components in terms of $\Psi(x,y)$. Note that $\Psi(x,y) \equiv \Psi(z,\bar{z})$. Then using eqn (6) yields

$$\frac{\partial\Psi}{\partial z} = \frac{1}{2}\left(\frac{\partial\Psi}{\partial x} - \mathrm{i}\frac{\partial\Psi}{\partial y}\right), \quad \frac{\partial\Psi}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial\Psi}{\partial x} + \mathrm{i}\frac{\partial\Psi}{\partial y}\right). \tag{35}$$

Thus using eqn (35) in eqn (12) gives the stress components as

$$\tau_{x3} = K\frac{\partial\Psi}{\partial x}, \quad \tau_{y3} = K\frac{\partial\Psi}{\partial y}. \tag{36}$$

Similarly, using eqns (35) and (19) in eqn (7) yields the strain components as

$$\gamma_{x3} = \left\{\frac{1}{4}\left[\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2\right]\right\}^{\frac{1-n}{2n}}\frac{\partial\Psi}{\partial x}, \quad \gamma_{y3} = \left\{\frac{1}{4}\left[\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2\right]\right\}^{\frac{1-n}{2n}}\frac{\partial\Psi}{\partial y}. \tag{37}$$

Next, note that

$$w(x,y) = \int_\Gamma \mathrm{d}w(x,y) = \int_\Gamma\left(\frac{\partial w}{\partial x}\mathrm{d}x+\frac{\partial w}{\partial y}\mathrm{d}y\right) = \int_\Gamma(\gamma_{x3}\,\mathrm{d}x+\gamma_{y3}\,\mathrm{d}y), \tag{38}$$

where the last equality is obtained from eqn (2). Then using eqn (37) in eqn (38) will finally give the displacement component as

$$w(x,y) = \int_\Gamma\left\{\frac{1}{4}\left[\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2\right]\right\}^{\frac{1-n}{2n}}\left(\frac{\partial\Psi}{\partial x}\mathrm{d}x+\frac{\partial\Psi}{\partial y}\mathrm{d}y\right), \tag{39}$$

where Γ, as the integration path, is a curve without singular points.

Once the harmonic function $\Psi(x,y)$ is determined from given boundary conditions, the stress, strain and displacement fields in a power-law hardening body under anti-plane shear will readily be calculated from eqns (36), (37) and (39).

3. ONE CLASS OF CLOSED-FORM SOLUTIONS

As a direct application of the general solution, we consider the simplest solution function $f(z)$. Consequently, we obtain from eqn (33b), by setting $D = 0 = E$:

$$f(z) = \frac{1}{\mathrm{i}C}\ln z. \tag{40}$$

In the cylindrical polar coordinates (r,θ), this becomes

$$f(z) = \frac{1}{\mathrm{i}C}(\ln r+\mathrm{i}\theta), \tag{41}$$

where

$$r = \sqrt{(x^2+y^2)}, \quad \theta = \tan^{-1}\frac{y}{x}. \tag{42}$$

Thus using eqns (41) and (42) in eqn (33a) will yield

$$\Psi(x,y) = \frac{1}{C}\ln\frac{1}{\sqrt{(x^2+y^2)}}. \tag{43}$$

Similarly, by setting $D_1 = 0 = E_1$ we obtain from eqn (34) that

$$\Psi(x,y)=\frac{1}{C_1}\ln\sqrt{(x^2+y^2)}. \tag{44}$$

Obviously, eqn (44) will be identical to eqn (43) if we take $C_1=-C$. Hence, we only consider the harmonic function given in eqn (43) in our analysis. This harmonic function provides one possible class of solutions for anti-plane problems of the power-law material. We show in the following that it does solve one type of such problems.

Consider a problem of the telescopic shear type, i.e. a long hollow cylinder ($R_i \leqslant r \leqslant R_o$) with its inner surface welded to a rigid cylinder and its outer surface subjected to a uniform longitudinal shear τ (force/unit area, $\tau > 0$). In the context of non-linear elasticity, this type of problem has been studied well [see e.g. Polignone and Horgan (1992)].

The boundary conditions are

$$\mathbf{t(n)}|_{r=R_o}=-\tau\,\mathbf{e}_3, \tag{45a}$$

$$w|_{r=R_i}=0, \tag{45b}$$

where $\mathbf{n}$ is the unit outward normal vector to the (outer) surface of the cylinder and $\mathbf{t(n)}$ is the traction vector associated with $\mathbf{n}$. Clearly, these are mixed-kind boundary conditions.

Suppose that $\Psi(x,y)$, given in eqn (43), will solve this problem. Then it follows from eqn (43) that

$$\frac{\partial\Psi}{\partial x}=-\frac{1}{C}\frac{x}{x^2+y^2},\quad \frac{\partial\Psi}{\partial y}=-\frac{1}{C}\frac{y}{x^2+y^2}. \tag{46}$$

Substituting eqn (46) into eqn (36) gives

$$\tau_{x3}=-\frac{K}{C}\frac{x}{x^2+y^2},\quad \tau_{y3}=-\frac{K}{C}\frac{y}{x^2+y^2}. \tag{47}$$

Note that for the present cylinder problem we have

$$\left.\begin{aligned}\boldsymbol{\sigma}&=\tau_{x3}(\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1)+\tau_{y3}(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2),\\ \mathbf{n}|_{r=R_o}&=(\cos\alpha\,\mathbf{e}_1+\sin\alpha\,\mathbf{e}_2)|_{r=R_o},\quad \mathbf{n}|_{r=R_i}=-(\cos\alpha\,\mathbf{e}_1+\sin\alpha\,\mathbf{e}_2)|_{r=R_i},\\ \cos\alpha&=\frac{x}{\sqrt{(x^2+y^2)}},\quad \sin\alpha=\frac{y}{\sqrt{(x^2+y^2)}},\end{aligned}\right\} \tag{48a–e}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, α is the angle between the x-axis and $\mathbf{n}$, and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are the base vectors of the Cartesian coordinate system. Then it follows from eqns (48a,b,d,e) that

$$\mathbf{t(n)}|_{r=R_o}=(\boldsymbol{\sigma}\mathbf{n})|_{r=R_o}=\left.\left(\tau_{x3}\frac{x}{\sqrt{(x^2+y^2)}}+\tau_{y3}\frac{y}{\sqrt{(x^2+y^2)}}\right)\right|_{r=R_o}\mathbf{e}_3. \tag{49}$$

Using eqns (47) and (49) in eqn (45a) will give

$$C = \frac{2^{n-1}}{(\sqrt{3})^{n+1}} \frac{A}{\tau R_o}. \tag{50}$$

Next, using eqn (46) in eqn (39) and carrying out the integration along the path $\Gamma: \theta =$ constant will give

$$w(r,\theta) = w(R_i,\theta) + 2\left(\frac{1}{2C}\right)^{1/n} \frac{n}{1-n}(r^{1-(1/n)} - R_i^{\,1-(1/n)}) \tag{51}$$

for $n \neq 1$. This can satisfy the remaining boundary condition, eqn (45b), for any C ($\neq 0$). Thus inserting eqn (50) into eqn (43) gives the solution function of this problem as

$$\Psi(x,y) = \frac{(\sqrt{3})^{n+1}}{2^{n-1}} \frac{\tau R_o}{A} \ln \frac{1}{\sqrt{(x^2+y^2)}}. \tag{52}$$

With $\Psi(x,y)$ so determined, the stress, strain and displacement components can easily be obtained from eqns (36), (37) and (39). Consequently, the stress components are obtained as

$$\tau_{x3} = -\tau R_o \frac{\cos\theta}{r}, \quad \tau_{y3} = -\tau R_o \frac{\sin\theta}{r}, \tag{53}$$

the strain components as

$$\left.\begin{aligned} \gamma_{x3} &= -(\sqrt{3})^{1+(1/n)}\left(\frac{\tau}{A}\right)^{1/n}\left(\frac{R_o}{r}\right)^{1/n}\cos\theta, \\ \gamma_{y3} &= -(\sqrt{3})^{1+(1/n)}\left(\frac{\tau}{A}\right)^{1/n}\left(\frac{R_o}{r}\right)^{1/n}\sin\theta, \end{aligned}\right\} \tag{54}$$

and the displacement component as

$$w(r,\theta) = (\sqrt{3})^{1+(1/n)} \frac{n}{1-n}\left(\frac{\tau}{A}\right)^{1/n}\left[r\left(\frac{R_o}{r}\right)^{1/n} - R_i\left(\frac{R_o}{R_i}\right)^{1/n}\right] \tag{55}$$

for $n \neq 1$, or

$$w(r,\theta) = \frac{3\tau R_o}{E} \ln \frac{R_i}{r} \tag{56}$$

for $n = 1$, $A \equiv E$, with E being Young's modulus. Equation (56) gives the displacement component of the elastic solution. Similarly, when $n = 1$, eqn (54) reduces to the strain components of the elastic solution, while eqn (53), which gives the stress components, remains the same for the present problem.

The other two slightly different problems of this type, i.e. the problem with mixed-kind boundary conditions

$$\mathbf{t}(\mathbf{n})|_{r=R_i} = \tau\,\mathbf{e}_3, \quad w|_{r=R_o} = 0, \tag{57}$$

and the problem with second-kind boundary conditions

$$w|_{r=R_i} = 0, \quad w|_{r=R_o} = \delta, \tag{58}$$

where $\delta(>0)$ is a real constant, can also be solved using the same harmonic function $\Psi(x, y)$ and following the same procedures as for the problem just solved above. Consequently, the solution of the first problem is found to be:

$$\left.\begin{aligned}
&\Psi(x,y) = \frac{(\sqrt{3})^{n+1}}{2^{n-1}}\frac{\tau R_i}{A}\ln\frac{1}{\sqrt{(x^2+y^2)}},\\
&\tau_{x3} = -\tau R_i\frac{\cos\theta}{r}, \quad \tau_{y3} = -\tau R_i\frac{\sin\theta}{r},\\
&\gamma_{x3} = -(\sqrt{3})^{1+(1/n)}\left(\frac{\tau}{A}\right)^{1/n}\left(\frac{R_i}{r}\right)^{1/n}\cos\theta, \quad \gamma_{y3} = -(\sqrt{3})^{1+(1/n)}\left(\frac{\tau}{A}\right)^{1/n}\left(\frac{R_i}{r}\right)^{1/n}\sin\theta,\\
&w(r,\theta) = (\sqrt{3})^{1+(1/n)}\frac{n}{1-n}\left(\frac{\tau}{A}\right)^{1/n}\left[r\left(\frac{R_i}{r}\right)^{1/n} - R_o\left(\frac{R_i}{R_o}\right)^{1/n}\right] \quad (n \neq 1),\\
&w(r,\theta) = \frac{3\tau R_i}{E}\ln\frac{R_o}{r} \quad (n = 1, A \equiv E).
\end{aligned}\right\} \tag{59}$$

When $n = 1$, eqn (59) gives the elastic solution of this problem.

The solution of the second problem is found to be:

$$\left.\begin{aligned}
&\Psi(x,y) = 2\left(\frac{\delta}{2}\right)^n\left(\frac{1}{n}-1\right)^n R_i^{1-n}\left[1-\left(\frac{R_o}{R_i}\right)^{1-(1/n)}\right]^{-n}\ln\frac{1}{\sqrt{(x^2+y^2)}},\\
&\tau_{x3} = -\frac{AR_i}{(\sqrt{3})^{n+1}}\left(\frac{1}{n}-1\right)^n\left(\frac{\delta}{R_i}\right)^n\left[1-\left(\frac{R_o}{R_i}\right)^{1-(1/n)}\right]^{-n}\frac{\cos\theta}{r},\\
&\tau_{y3} = -\frac{AR_i}{(\sqrt{3})^{n+1}}\left(\frac{1}{n}-1\right)^n\left(\frac{\delta}{R_i}\right)^n\left[1-\left(\frac{R_o}{R_i}\right)^{1-(1/n)}\right]^{-n}\frac{\sin\theta}{r},\\
&\gamma_{x3} = -\delta\left(\frac{1}{n}-1\right)\left\{R_i\left[1-\left(\frac{R_o}{R_i}\right)^{1-(1/n)}\right]\right\}^{-1}\left(\frac{R_i}{r}\right)^{1/n}\cos\theta,\\
&\gamma_{y3} = -\delta\left(\frac{1}{n}-1\right)\left\{R_i\left[1-\left(\frac{R_o}{R_i}\right)^{1-(1/n)}\right]\right\}^{-1}\left(\frac{R_i}{r}\right)^{1/n}\sin\theta,\\
&w(r,\theta) = \delta\frac{1-(r/R_i)^{1-(1/n)}}{1-(R_o/R_i)^{1-(1/n)}} \quad (n \neq 1), \quad w(r,\theta) = \delta\frac{\ln(r/R_i)}{\ln(R_o/R_i)} \quad (n = 1).
\end{aligned}\right\} \tag{60}$$

When $n = 1$, eqn (60) reduces, by a suitable limiting process, to the elastic solution identical to that given in Atkin and Fox (1980) for the same problem.

4. SUMMARY

The mathematical analysis presented here is conventional in contrast to those analyses based on the hodograph transformation technique, i.e. the roles of the independent and dependent variables are not interchanged in the present analysis. This analysis is made possible by using a complex variable representation and by introducing a differential transformation.

The formulation is for a power-law material (incompressible) and for infinitesimal deformations. Hencky's deformation theory and von Mises' yield criterion are used to represent the constitutive relations of the problem. However, for (incompressible) power-law materials with infinitesimal deformations and under monotonic proportional loading the use of an incremental theory would lead to the same results [see Ilyushin (1946)].

The general solution of the equivalent system derived in Section 2 is of a special type. This is due to the limitations of the differential transformation, but not the original system itself. In other words, the present analysis, being based on a differential transformation of a special property, can only lead to one particular type of solution of the original system. There can be other types of solutions which may also be derived by using analytical approaches, as the original system is a non-linear one and does not have any fixed solution form. However, whether such approaches are still conventional needs further investigation.

As a direct application of the general solution, we show that one class of closed-form solutions of the telescope shear type problem of the power-law material can easily be obtained in a unified way, which include their linear elastic counterparts as special cases. These solutions may alternatively be obtained by direct integration, as the telescope shear type problem considered is axisymmetric and is governed by a non-linear ordinary differential equation (of first order when expressed in terms of the stress component and using the polar coordinates). However, the procedures to be followed in the latter approach will be totally different from those presented here.

Acknowledgements—Most of this work was finished during July–August 1992, when the author was holding a summer research assistantship from the Department of Mechanical Engineering at the University of Alberta. This support is gratefully acknowledged. A comment from Dr P. Schiavone then has been helpful. The kindness and support of Dr P. Schiavone in the preparation of this paper are also greatly appreciated. The author is also indebted to one anonymous referee for helpful comments.

REFERENCES

Abeyaratne, R. and Horgan, C. O. (1983). Bounds on stress concentration factors in finite antiplane shear. *J. Elasticity* **13,** 49–61.
Adkins, J. E. (1954). Some generalizations of the shear problem for isotropic incompressible materials. *Proc. Cambr. Phil. Soc.* **50,** 334–345.
Amazigo, J. C. (1974). Fully plastic crack in an infinite body under anti-plane shear. *Int. J. Solids Structures* **10,** 1003–1015.
Atkin, R. J. and Fox, N. (1980). *An Introduction to the Theory of Elasticity*, pp. 157–158. Longman, London.
Boresi, A. P. and Chong, K. P. (1987). *Elasticity in Engineering Mechanics*, pp. 28–29. Elsevier, New York.
Eshelby, J. D. (1979). Boundary problems. In *Dislocations in Solids: The Elastic Theory* (Edited by F. R. N. Nabarro), Vol. 1, pp. 167–222. North-Holland, Amsterdam.
Flanigan, F. J. (1972). *Complex Variables, Harmonic and Analytic Functions*, pp. 148–193. Allyn and Bacon, Boston.
Gurtin, M. E. and Temam, R. (1981). On the anti-plane shear problem in finite elasticity. *J. Elasticity* **11,** 197–206.
Horgan, C. O. and Miller, K. L. (1994). Antiplane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. *ASME J. Appl. Mech.* **61,** 23–29.
Ilyushin, A. A. (1946). The theory of small elastic–plastic deformations. *Prikl. Mat. Mekh.* **10,** 347–356.
Jiang, Q. and Knowles, J. K. (1991). A class of compressible elastic materials capable of sustaining finite anti-plane shear. *J. Elasticity* **25,** 193–201.
Knowles, J. K. (1976). On finite anti-plane shear for incompressible elastic materials. *J. Aust. Math. Soc. Ser. B* **19,** 400–415.
Knowles, J. K. (1977a). A note on anti-plane shear for compressible materials in finite elastostatics. *J. Aust. Math. Soc. Ser. B* **20,** 1–7.
Knowles, J. K. (1977b). The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids. *Int. J. Fracture* **13,** 611–639.
Koskinen, M. F. (1963). Elastic–plastic deformation of a single grooved flat plate under longitudinal shear. *ASME J. Bas. Engng* **85,** 585–594.
Milne-Thomson, L. M. (1962). *Antiplane Elastic Systems*. Springer, Berlin.
Muskhelishvili, N. I. (1953). *Some Basic Problems of the Mathematical Theory of Elasticity*, pp. 129–134. P. Noordhoff, Groningen, Holland.
Neuber, H. (1961). Theory of stress concentration for shear-strained prismatical bodies with arbitrary nonlinear stress–strain law. *ASME J. Appl. Mech.* **28,** 544–550.
Polignone, D. A. and Horgan, C. O. (1992). Axisymmetric finite anti-plane shear of compressible nonlinearly elastic circular tubes. *Q. Appl. Math.* **50,** 323–341.
Raymond, J.-P. (1993). An anti-plane shear problem. *J. Elasticity* **33,** 213–231.
Rice, J. R. (1966). Contained plastic deformation near cracks and notches under longitudinal shear. *Int. J. Fracture Mech.* **2,** 426–447.
Rice, J. R. (1967). Stresses due to a sharp notch in a work-hardening elastic–plastic material loaded by longitudinal shear. *ASME J. Appl. Mech.* **34,** 287–298.
Tuba, I. S. (1969). A perturbation theory of antiplane elastic–plastic deformations. *Int. J. Non-Linear Mech.* **4,** 51–60.